MATH 251, Mariano Echeverria

Problems Integration of Scalar Fields

This material corresponds roughly to sections 15.1, 15.2, 15.3, 12.7, 15.4, 15.6 and 16.4 in the book.

Problem 1. Consider

$$I = \int\int_R \cos\sqrt{y-x}\, dA \tag{1}$$

where R **is the region determined by the curves** $y = x + 1$, $y = x^2 + x$. **Find** I **using the change of variables** $u = x$, $v = \sqrt{y-x}$

The region of integration on the xy plane is

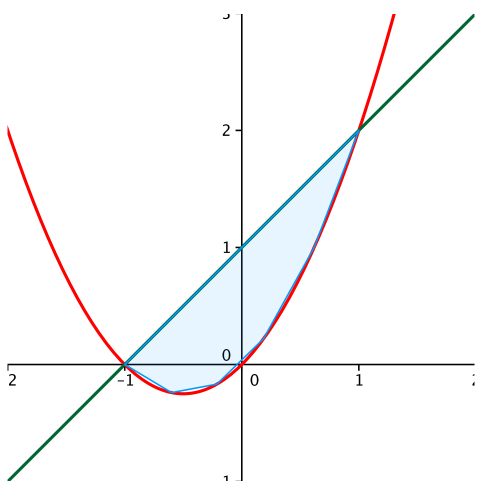

Figure 1: Region of integration xy plane

We have to find how the region transforms under the change of variables. Notice that

$$v^2 = y - x \tag{2}$$

so

$$y = v^2 + x = v^2 + u \tag{3}$$

which means that the straight line

$$y = x + 1 \tag{4}$$

becomes

$$v^2 + u = u + 1 \tag{5}$$

In other words, we get

$$v^2 = 1 \tag{6}$$

which implies

$$v = 1 \tag{7}$$

since $v = \sqrt{y - x} \geq 0$.

The parabola $y = x^2 + x$ becomes $v^2 + u = u^2 + u$ or $v = \pm u$. Therefore the region of integration with respect to the uv plane

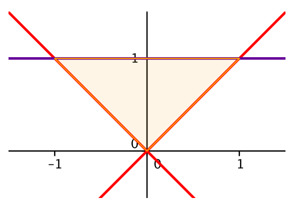

Figure 2: Region of integration uv plane

The Jacobian of this change of variables is

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 2v \end{vmatrix} = 2v \tag{8}$$

so the integral is

$$I = \int_0^1 \left(\int_{-v}^{v} \cos(v)\, 2v du \right) dv = 8\cos(1) - 4\sin(1) \tag{9}$$

where integration by parts was used.

Problem 2. Find the volume of the solid of revolution given by the equation $z^2 \geq x^2 + y^2$, which is contained inside the sphere $x^2 + y^2 + z^2 = 1$

The surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = 1$ are shown in the next figure

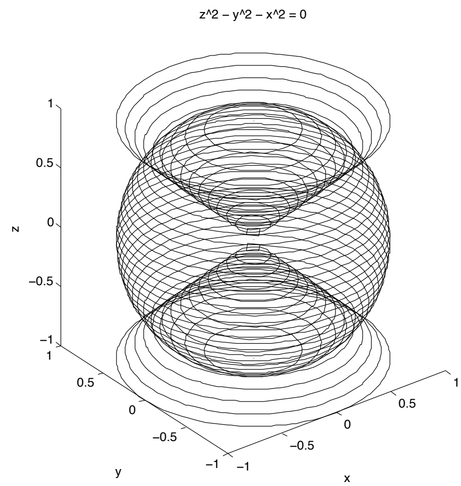

Figure 3: Sphere-cone

We will use spherical coordinates, where the equations become $\cos^2\theta = \sin^2\theta$, $r = 1$ [recall that θ is the angle measured with respect to the z axis]. By symmetry with respect to the angle φ we can look at a cross section [the xz plane] to find the limits with respect to r, θ.

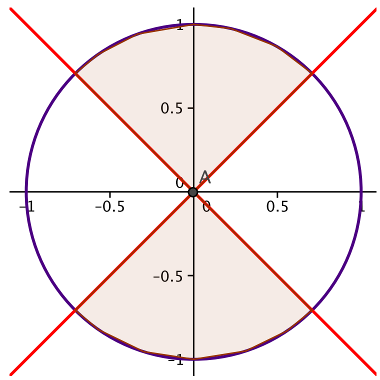

Figure 4: Intersection of the sphere-cone with the plane $y = 0$

From this figure we can see that the limits of integration are $0 \leq \varphi < 2\pi$, $0 \leq \theta \leq \frac{\pi}{4}$, $0 \leq r \leq 1$. Hence the volume is

$$V = 2\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 r^2 \sin\theta dr d\theta d\varphi = \frac{4\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right) \tag{10}$$

Problem 3. Prove Newton's Shell theorem for the gravitational potential. Namely, the gravitational potential created by an object with constant density ρ_M and spherically shaped on a point $(0, 0, z_0)$ is

$$\int\int\int \frac{-Gdm}{\sqrt{x^2 + y^2 + (z - z_0)^2}} \tag{11}$$

where, $dm = \rho_M dvol$, G is Newton's universal gravitational constant, and the region of integration is the interior of the sphere of radius R centered at the origin. Use spherical coordinates to show that this integral equals

$$\begin{cases} -\frac{2}{3}\pi G\rho_M \left(3R^2 - z_0^2\right) & \textbf{if } 0 < z_0 \leq r \\ -\frac{GM}{z_0} & \textbf{if } r < z_0 \end{cases} \tag{12}$$

We use spherical coordinates. Since $dm = \rho_M dV$ we must compute

$$-G\rho_M \int_0^{2\pi}\int_0^{\pi}\int_0^R \frac{r^2 \sin\theta dr d\theta d\varphi}{\sqrt{r^2\sin^2\theta + (r\cos\theta - z_0)^2}} \tag{13}$$

The integrand does not depend on φ so we can integrate this variable first. We can also change the order of integration and find

$$-2\pi G\rho_M \int_0^R\int_0^{\pi} \frac{r^2 \sin\theta d\theta dr}{\sqrt{r^2 - 2rz_0\cos\theta + z_0^2}} \tag{14}$$

Now we make the change of variables $u = r^2 - 2rz_0\cos\theta + z_0^2$, $du = 2rz_0 \sin\theta d\theta$ and we end up integrating

$$-\frac{\pi}{z_0}G\rho_M \int_0^R \int_{(r-z_0)^2}^{(r+z_0)^2} \frac{rdudr}{\sqrt{u}} = -\frac{2\pi}{z_0}G\rho_M \int_0^R \left(|r+z_0| - |r-z_0|\right) rdr \tag{15}$$

We may assume $z_0 \geq 0$ so we must analyze the cases $0 \leq z_0 \leq r$, $r < z_0$.

If $0 \leq z_0 \leq r$ then

$$-\frac{2\pi}{z_0}G\rho_M \left(\int_0^{z_0} (r + z_0 - (z_0 - r))\, rdr + \int_{z_0}^R (r + z_0 - (r - z_0))\, rdr\right) \tag{16}$$

$$= -\frac{4\pi}{z_0}G\rho_M \left(\int_0^{z_0} r^2 dr + \int_{z_0}^R z_0 rdr\right) = -\frac{4\pi}{z_0}G\rho_M \left(\frac{z_0^3}{3} + \frac{z_0}{2}\left(R^2 - z_0^2\right)\right) \tag{17}$$

$$= -\frac{2}{3}\pi G\rho_M \left(3R^2 - z_0^2\right) \tag{18}$$

If $r < z_0$ then

$$-\frac{4\pi}{z_0}G\rho_M \int_0^R r^2 dr = -\frac{4\pi}{z_0}G\rho_M \frac{R^3}{3} = -\frac{GM}{z_0} \tag{19}$$

Problem 4. Consider the Gaussian integral

$$I_a = \int\int_D e^{-\left(x^2+y^2\right)} dxdy \tag{20}$$

where D is the disk $x^2 + y^2 \leq a^2$.

a) Use polar coordinates to show that $I_a = \pi\left(1 - e^{-a^2}\right)$.

b) Find $\int_0^\infty e^{-x^2}dx$ using the value of $\int\int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy$.

a) In polar coordinates we find that

$$I_a = \int_0^{2\pi}\left(\int_0^a e^{-r^2} rdrd\theta\right) = 2\pi \int_0^a e^{-r^2} rdr \tag{21}$$

Using the change of variables $u = -r^2$, $du = -2rdr$ we must compute

$$= 2\pi\left(-\frac{1}{2}\right)\int_0^{-a^2} e^u du = -\pi\ e^u\big|_0^{-a^2} = \pi\left(1 - e^{-a^2}\right) \tag{22}$$

b) Observe that

$$\int\int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}dydx \tag{23}$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 \tag{24}$$

Taking the limit $a \longrightarrow \infty$ in part a) we obtain

$$\int\int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy = \pi \tag{25}$$

so

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{26}$$

Example 5. Find the average value of the temperature $T(x, y, z) = x^2 + y^2 - z^2$ inside the interior of the region bounded by the surfaces $2z = x^2 + y^2$, $x^2 + y^2 - z^2 = 1$ and $z = 0$, $z = 3$. You can use that the average value of T, denoted $< T >$, is given by

$$< T > = \frac{\int\int\int_R T dV}{\mathbf{Vol}(R)} \tag{27}$$

Using cylindrical coordinates the equations of the surfaces are $2z = r^2$, $r^2 = 1 + z^2$.

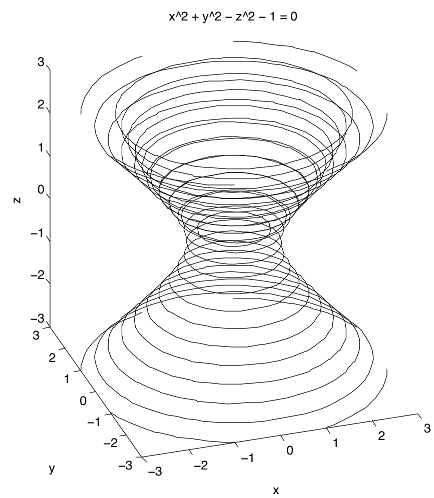

Figure 5: Paraboloid-hyperboloid

On the xz plane the cross sections of the surfaces are

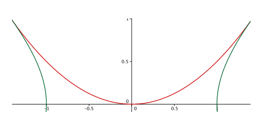

Figure 6: Intersection paraboloid-hyperboloid with the xz plane

The limits of integration in cylindrical coordinates become

$$0 \le \theta \le 2\pi \quad 0 \le z \le 1 \quad \sqrt{2z} \le r \le \sqrt{1+z^2} \tag{28}$$

First we compute the volume

$$V = \int_0^{2\pi} \int_0^1 \int_{\sqrt{2z}}^{\sqrt{1+z^2}} r dr dz d\theta = \int_0^{2\pi} \int_0^1 \left. \frac{r^2}{2} \right|_{\sqrt{2z}}^{\sqrt{1+z^2}} dz d\theta \tag{29}$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^1 (z-1)^2 \, dz d\theta = \frac{1}{6} \int_0^{2\pi} \left. (z-1)^3 \right|_0^1 d\theta = \frac{\pi}{3} \tag{30}$$

We also need to compute

$$\int\int\int T dV = \int_0^{2\pi} \int_0^1 \int_{\sqrt{2z}}^{\sqrt{1+z^2}} r \left(r^2 - z^2\right) dr dz d\theta = 2\pi \int_0^1 \left. \left(\frac{r^4}{4} - z^2 \frac{r^2}{2} \right) \right|_{\sqrt{2z}}^{\sqrt{1+z^2}} dz \tag{31}$$

$$= 2\pi \int_0^1 \left(\frac{\left(1+z^2\right)^2}{4} - z^2 \frac{(1+z^2)}{2} - \frac{(2z)^2}{4} + z^2 \frac{(2z)^2}{2} \right) dz \tag{32}$$

$$= \frac{\pi}{2} \int_0^1 \left(1 + 2z^2 + z^4 - 2z^2 - 2z^4 - 4z^2 + 8z^4\right) dz = \frac{\pi}{2} \int_0^1 \left(7z^4 - 4z^2 + 1\right) dz = \frac{\pi}{2} \left(\frac{7}{5} - \frac{4}{3} + 1 \right) = \frac{16\pi}{30} \tag{33}$$

Therefore the average value

$$\langle T \rangle = \frac{16\pi}{10} \tag{34}$$

Problem 6. Consider the region R determined by the surfaces $z = \sqrt{x^2+y^2}$, $z = 2 - x^2 - y^2$. Write the integral for the volume of this region using cylindrical coordinates, first using the order of integration $dzdrd\theta$, and then using $drdzd\theta$. You do not need to compute the value of the integral!

The surfaces are shown in the following figure

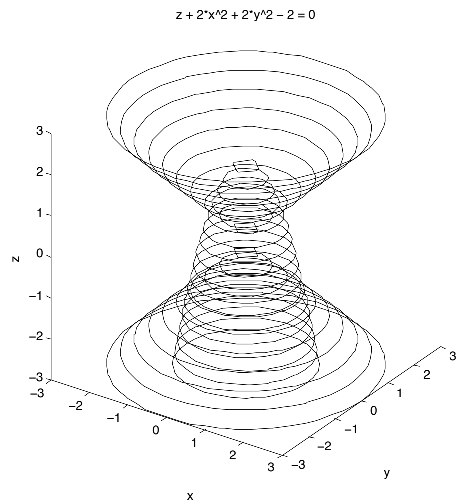

Figure 7: Cone-Paraboloid

Using cylindrical coordinates the equations are $z = r$, $z = 2 - r^2$. With respect to the xz plane the cross section looks like

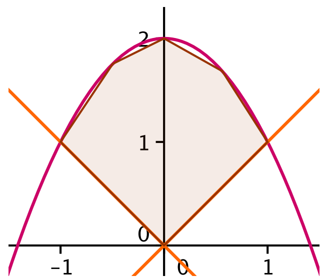

Figure 8: Intersection cone-paraboloid with the xz plane

Both surfaces intersect when $r = 2 - r^2$, that is, $z = r = 1$. To find the integral in the order $dzdrd\theta$ notice that

$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 1 \quad r \leq z \leq 2 - r^2 \tag{35}$$

So the volume is

$$V = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r dz dr d\theta \tag{36}$$

To find the integral in the order $drdzd\theta$ notice that

$$0 \leq \theta \leq 2\pi \quad 0 \leq z \leq 2 \tag{37}$$

In this case the r bounds depend on z

$$\begin{cases} 0 \leq z \leq 1 & 0 \leq r \leq z \\ 1 \leq z \leq 2 & 0 \leq r \leq \sqrt{2-z} \end{cases} \tag{38}$$

Therefore

$$V = \int_0^{2\pi} \int_0^1 \int_0^z r dz dr d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} r dz dr d\theta \tag{39}$$

Problem 7. Consider the double integral $I = \int_0^{\pi} \int_{\sin x}^{3+\cos(2x)} f(x,y) dy dx$.
a) Draw the region of integration R.
b) Change the order of integration to $dxdy$. Do not compute the integral.

a) We have $0 \leq x \leq \pi$, $\sin x \leq y \leq 3 + \cos(2x)$. The region of integration is

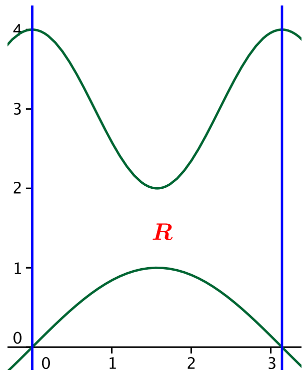

Figure 9: Region of integration

b) To change the bounds notice first of all that $0 \leq y \leq 4$. For the bounds in x we must break the region into the pieces determined by the inequalities $0 \leq y \leq 1$, $1 \leq y \leq 2$, $2 \leq y \leq 4$.

We also use the fact that $\sin(\pi - \alpha) = \sin(\alpha)$, $\cos(2\pi - \alpha) = \cos(\alpha)$. If we consider $\arcsin \alpha$ as a function with domain between 0 and $\frac{\pi}{2}$ we must have

$$0 \leq y \leq 1 \quad 0 \leq x \leq \arcsin y \quad \pi - \arcsin y \leq x \leq \pi \tag{40}$$

$$1 \leq y \leq 2 \quad 0 \leq x \leq \pi \tag{41}$$

$$2 \leq y \leq 4 \quad 0 \leq x \leq \frac{1}{2}\arccos(y-3) \quad \pi - \frac{1}{2}\arccos(y-3) \leq x \leq \pi \tag{42}$$

Therefore the integral is

$$\int_0^1 \int_0^{\arcsin y} f dxdy + \int_0^1 \int_0^{\arcsin y} f dxdy + \int_1^2 \int_0^{\pi} f dxdy + \int_2^4 \int_0^{\frac{\arccos(y-3)}{2}} f dxdy + \int_2^4 \int_{\pi - \frac{\arccos(y-3)}{2}}^{\pi} f dxdy \tag{43}$$

Problem 8. Consider the integral $I = \int\int\int_T \frac{xy+y^2}{x^3} dxdydz$, where T is the region inside the first octant $(x, y, z \geq 0)$ between the plane $x+y+z=2$, the xy plane, and the vertical "walls" determined by the trapezoid given by the equations $x+y=1$, $x+y=2$, $y=0$, $y=x$. As a suggestion, use the change of variables $x = \frac{v}{1+w}$, $y = \frac{vw}{1+w}$, $z = u - v$.

First of all the plane $x+y+z=2$ intersects the plane xy when $z=0$, that is, when $x+y=2$. Therefore on the xy plane the region of integration looks like

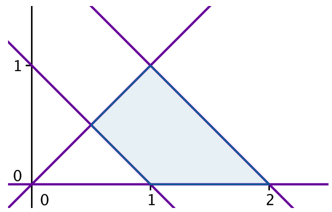

Figure 10: Trapezoid

Using the change of variables $x = \frac{v}{1+w}$, $y = \frac{vw}{1+w}$, $z = u - v$ the line $x + y = 1$ becomes $\frac{v}{1+w} + \frac{vw}{1+w} = 1$, that is, $v = 1$.

Similarly, the line $x + y = 2$ becomes $v = 2$.

The line $y = 0$ becomes $\frac{vw}{1+w} = 0$, observe that $v \neq 0$ since $x \neq 0$ so the bound corresponds to the line $w = 0$.

Similarly, the line $y = x$ becomes $\frac{v}{1+w} = \frac{vw}{1+w}$ or $w = 1$.

Finally, the plane $z = 0$ becomes $u = v$, while the plane $x + y + z = 2$ becomes $\frac{v}{1+w} + \frac{vw}{1+w} + u - v = 2$ or $u = 2$.

Therefore the bounds end up being

$$1 \leq v \leq 2 \quad 0 \leq w \leq 1 \quad v \leq u \leq 2 \tag{44}$$

Now we compute the Jacobian

$$J(u, v, w) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{1+w} & -\frac{v}{(1+w)^2} \\ 0 & \frac{w}{1+w} & v\left(\frac{1+w-w}{(1+w)^2}\right) \\ 1 & -1 & 0 \end{vmatrix} \tag{45}$$

$$= \frac{v}{(1+w)^3} \begin{vmatrix} 0 & 1 & -1 \\ 0 & w & 1 \\ 1 & -1 & 0 \end{vmatrix} = \frac{v}{(1+w)^2} \tag{46}$$

At the same time,

$$\frac{xy + y^2}{x^3} = \frac{y(x+y)}{x^3} = \frac{\frac{vw}{1+w}(v)}{\left(\frac{v}{1+w}\right)^3} = \frac{w(1+w)^2}{v} \tag{47}$$

Therefore the integral we must compute is

$$\int_1^2 \int_0^1 \int_v^2 w du dw dv = \int_1^2 \int_0^1 w(2 - v) dw dv = \int_1^2 (2 - v) \left. \frac{w^2}{2} \right|_0^1 dv \tag{48}$$

$$= \frac{1}{2} \int_1^2 (2 - v) dv = \frac{1}{2} \left. \left(2v - \frac{v^2}{2}\right) \right|_1^2 = \frac{1}{2} \left(2 - \frac{3}{2}\right) = \frac{1}{4} \tag{49}$$

Problem 9. Consider the region determined by the surfaces $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 = 3z$.

a) Write an integral for the volume of this region using cylindrical coordinates, using the order $dzdrd\theta$.

b) Write the same integral now using spherical coordinates in the order $drd\varphi d\theta$, where φ represents the angle which starts from the z axis.

The region of integration corresponds to

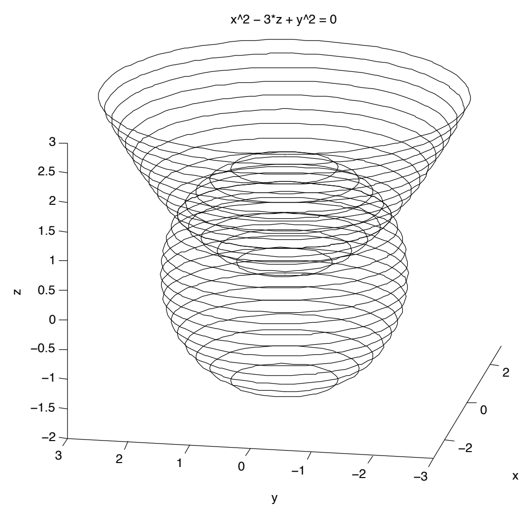

Figure 11: Region of integration paraboloid-sphere

a) With respect to the cylindrical coordianates $x = r\cos\theta$, $y = r\sin\theta$ the equation of the sphere becomes $r^2 + z^2 = 4$, while the paraboloid can be written as $r^2 = 3z$. A cross section of these surfaces is

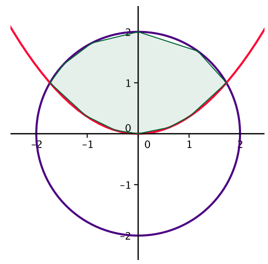

Figure 12: Region of integration on the xz plane

Using the order $dzdrd\theta$ we have

$$0 \leq \theta \leq 2\pi \tag{50}$$

The surfaces intersect when $r^2 + \frac{1}{9}r^4 = 4$, that is $r = \sqrt{3}$. Therefore

$$0 \leq r \leq \sqrt{3} \tag{51}$$

Finally,

$$\frac{r^2}{3} \leq z \leq \sqrt{4 - r^2} \tag{52}$$

In this way the volume becomes

$$V = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{\frac{r^2}{3}}^{\sqrt{4-r^2}} rdzdrd\theta \tag{53}$$

b) Using spherical coordinates $x = r\sin\varphi\cos\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\varphi$, with respect to the order $drd\varphi d\theta$ we must have

$$0 \leq \theta \leq 2\pi \tag{54}$$

From the previous figure we find that

$$0 \leq \varphi \leq \frac{\pi}{2} \tag{55}$$

To find the bounds for r the surfaces in spherical coordinates can be written as $r = 2$, $r^2\sin^2\varphi = 3r\cos\varphi$ or $r = \frac{3\cos\varphi}{\sin^2\varphi}$.

At the same time the surfaces intersect when $2 = \frac{3\cos\varphi}{\sin^2\varphi}$. This last equation can be rewritten as $2\left(1 - \cos^2\varphi\right) = 3\cos\varphi$ or $2\cos^2\varphi + 3\cos\varphi - 2 = 0$, which we rewrite as $(2\cos\varphi - 1)(\cos\varphi + 2) = 0$. Thus $\cos\varphi = \frac{1}{2}$ or $\cos\varphi = -2$, and since the last one is impossible we conclude that $\varphi = \frac{\pi}{3}$. Therefore the bounds of integration

$$\begin{cases} 0 \leq \varphi \leq \frac{\pi}{3} & 0 \leq r \leq 2 \\ \frac{\pi}{3} \leq \varphi \leq \frac{\pi}{2} & 0 \leq r \leq \frac{3\cos\varphi}{\sin^2\varphi} \end{cases} \tag{56}$$

so the volume is

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^{2} r^2 \sin\varphi dr d\varphi d\theta + \int_0^{2\pi} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{\frac{3\cos\varphi}{\sin^2\varphi}} r^2 \sin\varphi dr d\varphi d\theta \tag{57}$$

Problem 10. Make the change of variables $u = xy$, $v = \frac{y}{x}$ to find the volume of the solid bounded by the surfaces $z = x + y$, $xy = 1$, $xy = 2$, $y = x$, $y = 2x$, $z = 0$ $(x > 0,\ y > 0)$.

The region of integration on the xy plane is

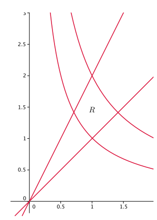

Figure 13: Region of integration xy plane

We must do the integral

$$\int\int_R (x + y)\, dy dx \tag{58}$$

With respect to the change of variables we have

$$1 \leq u \leq 2 \quad 1 < v < 2 \tag{59}$$

The Jacobian has the property that $J(u, v) = \frac{1}{J(x,y)}$ where

$$J(x, y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{2y}{x} = 2v \tag{60}$$

In this way

$$J(u, v) = \frac{1}{2v} \tag{61}$$

Since $y=\sqrt{uv}$, $x=\sqrt{\frac{u}{v}}$ the integral becomes

$$\int_1^2\int_1^2\left(\sqrt{\frac{u}{v}}+\sqrt{uv}\right)\frac{1}{2v}dudv=\int_1^2\left(v^{-\frac{3}{2}}+v^{-\frac{1}{2}}\right)\left.\frac{u^{\frac{3}{2}}}{3}\right|_1^2 dv \tag{62}$$

$$=\frac{1}{3}\left(2\sqrt{2}-1\right)\int_1^2\left(v^{-\frac{3}{2}}+v^{-\frac{1}{2}}\right)dv=\frac{1}{3}\left(2\sqrt{2}-1\right)\left.\left(-2v^{-\frac{1}{2}}+2v^{\frac{1}{2}}\right)\right|_1^2=\frac{2}{3}\left(2\sqrt{2}-1\right)\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right) \tag{63}$$

Problem 11. Consider the following triple integral

$$I=\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}}\int_0^{\sqrt{1-x^2-y^2}}f(x,y,z)dzdydx \tag{64}$$

a) Draw the region of integration.
b) Write I in spherical coordinates, using the order of integration $drd\varphi d\theta$, and the order $d\varphi drd\theta$. In both cases θ is the angle that starts from the x axis.

a) The bounds are $-\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$, $-\sqrt{\frac{1}{2}-x^2}\le y\le\sqrt{\frac{1}{2}-x^2}$, $0\le z\le\sqrt{1-x^2-y^2}$, and the region of integration is

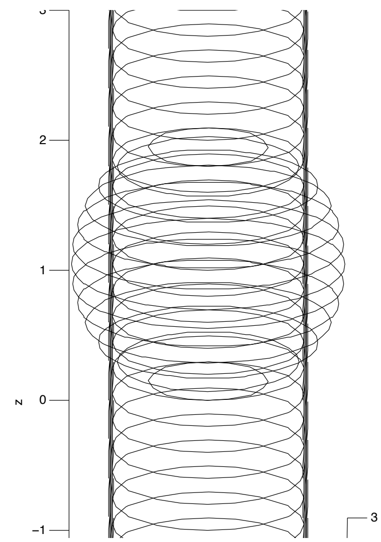

Figure 14: Sphere-cylinder-plane

With respect to spherical coordinates $x = r\sin\varphi\cos\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\varphi$ the bounds of the integral can be written as

$$r^2\sin^2\varphi = \frac{1}{2} \quad r = 1 \tag{65}$$

So a cross section looks like

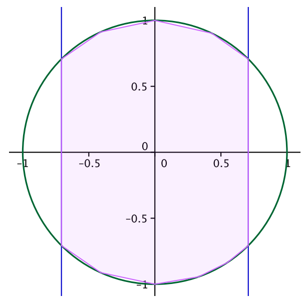

Figure 15: Cross section sphere-cylinder

Using the order $drd\varphi d\theta$ notice that

$$0 \leq \theta \leq 2\pi \tag{66}$$

Moreover,

$$0 \leq \varphi \leq \frac{\pi}{2} \tag{67}$$

The cylinder and sphere intersect when

$$\sin^2\varphi = \frac{1}{2} \tag{68}$$

so

$$\varphi = \frac{\pi}{4} \tag{69}$$

Therefore the bounds on r are

$$\begin{cases} 0 \leq \varphi \leq \frac{\pi}{4} & 0 \leq r \leq 1 \\ \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2} & 0 \leq r \leq \frac{1}{\sqrt{2}\sin\varphi} \end{cases} \tag{70}$$

and the integral can be written as

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 fr^2\sin\varphi drd\varphi d\theta + \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{1}{\sqrt{2}\sin\varphi}} fr^2\sin\varphi drd\varphi d\theta \tag{71}$$

With respect to the order $d\varphi drd\theta$ we have

$$0 \leq \theta \leq 2\pi \tag{72}$$

and the bounds for r are

$$0 \leq r \leq 1 \tag{73}$$

To find φ in terms of r, imagine that we fix a sphere of radius r, which in the next figure is represented as a circle of radius r.

There are two important cases to consider

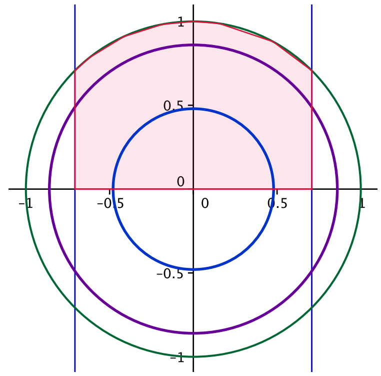

Figure 16: A couple of circles in the cross section for the sphere-cylinder

The first case is when the radius is less or equal to $\frac{1}{\sqrt{2}}$ (which corresponds to the blue circle). Here

$$0 \leq r \leq \frac{1}{\sqrt{2}} \qquad 0 \leq \varphi \leq \frac{\pi}{2} \tag{74}$$

The second case corresponds to the radius being between $\frac{1}{\sqrt{2}}$ and 1. Here

$$\frac{1}{\sqrt{2}} \leq r \leq 1 \qquad 0 \leq \varphi \leq \arcsin\left(\frac{1}{\sqrt{2}r}\right) \tag{75}$$

Therefore the integral is

$$\int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_0^{\frac{\pi}{2}} f r^2 \sin\varphi dr d\varphi d\theta + \int_0^{2\pi} \int_{\frac{1}{\sqrt{2}}}^{1} \int_0^{\arcsin\left(\frac{1}{\sqrt{2}r}\right)} f r^2 \sin\varphi dr d\varphi d\theta \tag{76}$$